Generating families on Lagrangian cobordisms

International Symposium on Natural Sciences at INU

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USC

Legendrians and Lagrangian cobordisms

Definition

A contact manifold (Y, ξ) is a $(2n+1)$-dimensional manifold with a maximally non-integrable hyperplane distribution ξ. A Legendrian submanifold $\Lambda \subset (Y, \xi)$ is an n-dimensional submanifold such that $T\Lambda \subset \xi$.

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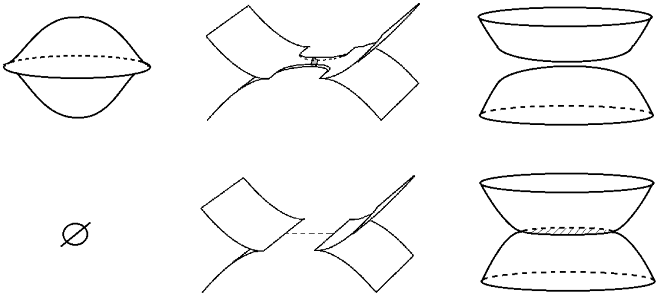
Definition

Let $\Lambda_-, \Lambda_+ \subset (Y, \ker\alpha)$ be Legendrian submanifolds. An exact Lagrangian cobordism L from Λ_- to Λ_+ is an exact Lagrangian $L \subset (\mathbb{R}_+ \times Y, d(s\alpha))$, such that

- for s_- small, $L \cap (0, s_-) \times Y = (0, s_-) \times \Lambda_-$;
- for s_+ large, $L \cap (s_+, +\infty) \times Y = (s_+, +\infty) \times \Lambda_+$;
- the primitive f_L such that $df_L = s\alpha|_L$ is a constant on $(0, s_-) \times \Lambda_-$ and on $(s_+, +\infty) \times \Lambda_+$.

Legendrians and Lagrangian cobordisms

- A Lagrangian cobordism from the empty set to a Legendrian is called a Lagrangian filling.



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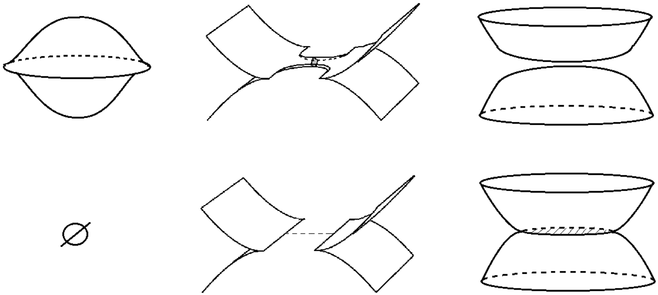

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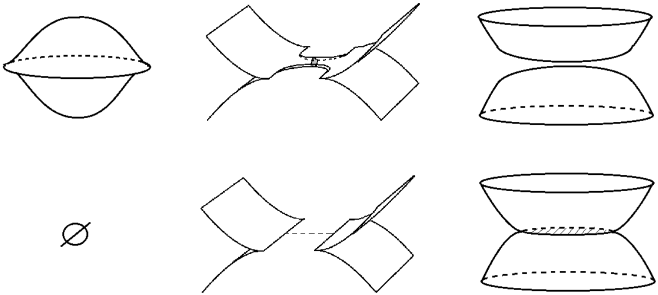
- There is a Lagrangian k-handle cobordism $0 \leq k \leq \dim \Lambda - 1$ (Dimitroglou Rizell '13).



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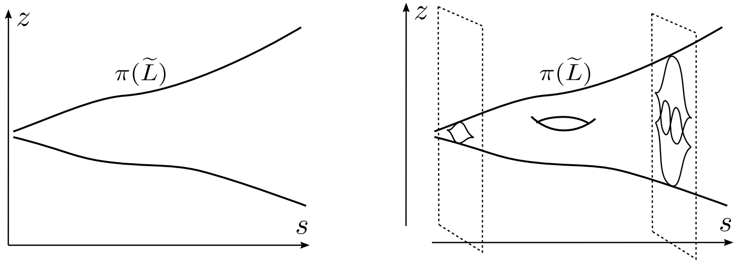

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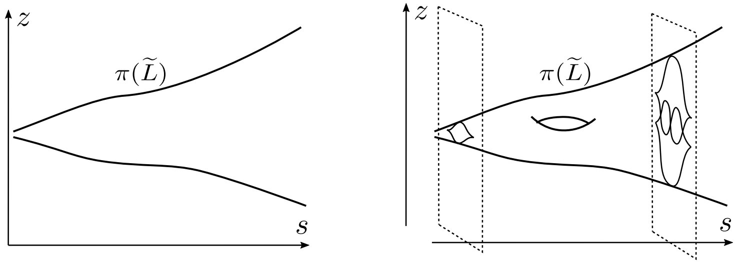

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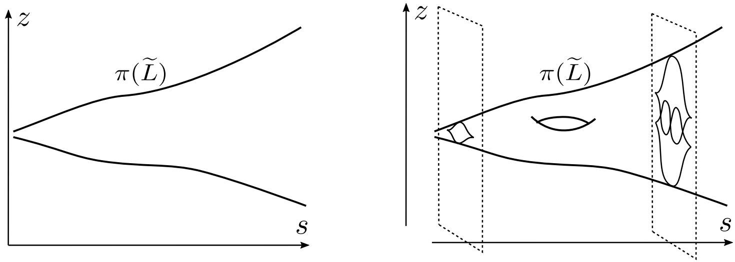
- The exact Lagrangian cobordism $L \subset \mathbb{R}_+ \times J^1M \cong T^*(M \times \mathbb{R}_+)$ lifts to a Legendrian cobordism in $J^1(M \times \mathbb{R}_+)$ (with conical boundary conditions, indicating the nonsymmetry).



Generating family of functions

- $\Lambda_{j^1 f} = \{(x, df(x), f(x)) \mid x \in M\}$ is a simple class of Legendrians in $J^1 M$. The front projection of $\pi(\Lambda_{j^1 f}) \subset M \times \mathbb{R}$ is the graph of the function f.

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- For general Legendrians in $J^1 M$, we cannot represent them as graphs of functions. Instead we can only view them as multivalued functions.
- We can add extra dimensions to M to resolve multivalued functions into genuine functions and then project the graph onto $M \times \mathbb{R}$.
- Consider a function $F : M \times \mathbb{R}^N \to \mathbb{R}$.

$$\Lambda_{j^1 F} = \{(x, u, \partial_x F, \partial_u F, F) \mid (x, u) \in M \times \mathbb{R}^N\}.$$

We want to project it to $\{(x, \partial_x F, F)\} = \Lambda \subset J^1 M$, but there are too many variables. Therefore, we require $\partial_u F = 0$, so there are only finitely many u values in the preimage.

Definition (Germ of generating families)

Let $U \subset M \times \mathbb{R}^N$ and $F : U \to \mathbb{R}$. Then (U, F) is called a germ of generating families for $\Lambda \subset J^1M$ if all fiberwise critical points $\partial_u F(x, u) = 0$ are nondegenerate and

$$\Lambda = \{(x, \partial_x F(x, u), F(x, u)) \mid (x, u) \in U\}.$$

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Definition

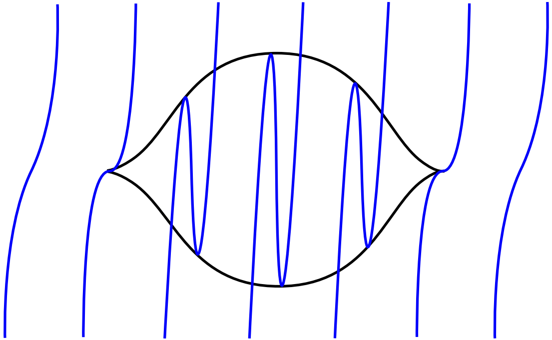
Let $F : M \times \mathbb{R}^N \to \mathbb{R}$. Then F is called a generating family (linear at infinity) for $\Lambda \subset J^1 M$ if all fiberwise critical points $\partial_u F(x, u) = 0$ are nondegenerate,

$$\Lambda = \{(x, \partial_x F(x, u), F(x, u)) \mid (x, u) \in M \times \mathbb{R}\}$$

and $F(x, u) = L_x(u)$ outside a proper subset $K \subset M \times \mathbb{R}^N$ where L_x is a linear function for any $x \in M$.

Example

There exists a generating function $F : \mathbb{R}_x \times \mathbb{R}_u \to \mathbb{R}$ for the Legendrian unknot in $J^1\mathbb{R}_x$ given by $F(x, u) = u^3 - \rho(x)u$, where $\rho(x)$ is positive in $(-1, 1)$ and negative outside $(-1, 1)$. (Jordan–Traynor '06 proved that it is unique up to stabilizations by extra quadratic factors.)



Morse homology of generating families

Theorem (Sabloff–Traynor '13)

Let L be an exact Lagrangian cobordism from Λ_- to Λ_+ in J^1M. Suppose L has a generating family linear at infinity that restricts to f_- and f_+ at the two ends, then there is a long exact sequence between Morse cohomologies

$$\cdots \to H^k(L, \Lambda_-) \to HM^k(f_-) \to HM^k(f_+) \to \dots.$$

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$$\cdots \to H^k(L, \Lambda_+) \to HM^k(f_-) \to HM_{n-k}(f_+) \to \dots.$$

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- **Question:** When can we find a generating family on an exact Lagrangian cobordism?
- **1.** We know Legendrians that do not have generating families, this will imply that the trivial cobordism of these Legendrians cannot support generating families.
- **2.** We hope that generating families on Λ_- can be extended to L, and hence can be restricted to Λ_+.
- The direction goes from Λ_- to Λ_+ because augmentations of the Legendrian contact dg algebra on Λ_- induce augmentations on Λ_+, and in microlocal sheaves, sheaves microsupported on Λ_- induce sheaves microsupported on Λ_+ once the local system data extends.

Theorem (Giroux '88, Latour '91)

$L \subset J^1M$ supports a germ of generating families if and only if the Lagrangian Gauss map $L \to U/O$ is null homotopic.
Moreover, germs of generating families on L up to local diffeomorphisms and stabilization by quadratic forms are classified by homotopy classes of the null homotopies, i.e., homotopy classes $L \to \Omega(U/O) \xrightarrow{\sim} \mathbb{Z} \times BO$.

Corollary (extension of germs of generating families)

For $L \subset J^1M$ and $L_0 \subset L$ an open submanifold (with smooth boundary), a germ of generating families on L_0 extends to L if and only if $L \to U/O$ is null homotopic, and the given null homotopy of $L_0 \to U/O$ extends to L.
(Note that no directionality is involved.)

Theorem (L., extension of generating families linear at infinity)

For $L \subset J^1M$ a Legendrian cobordism from Λ_- to Λ_+ with no Reeb chords, a generating family linear at infinity on Λ_- extends to L (and then restricts to Λ_+) if and only if $L \to U/O$ is null homotopic, and the given null homotopy of $\Lambda_- \to U/O$ extends to L.

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Remark

1. *A generating family linear at infinity can always be extended to an exact Lagrangian concordance.*
2. *A Lagrangian filling of Legendrian in J^1M with trivial Lagrangian Gauss map always induces a generating family linear at infinity.*
3. *For Legendrian knots and Lagrangian surface cobordisms, the condition is reduced to (1) the Maslov class of L vanishes and (2) the Maslov potential on Λ_- extends to L.*

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- **3.** Nearby Lagrangians in T^*M with trivial Lagrangian Gauss map (Abouzaid–Courte–Guillermou–Kragh '21), and nearby Lagrangians in T^*S^n and in particular Lagrangian fillings of the standard Legendrian unknot (Kragh '13, Abouzaid–Courte–Guillermou–Kragh '21);

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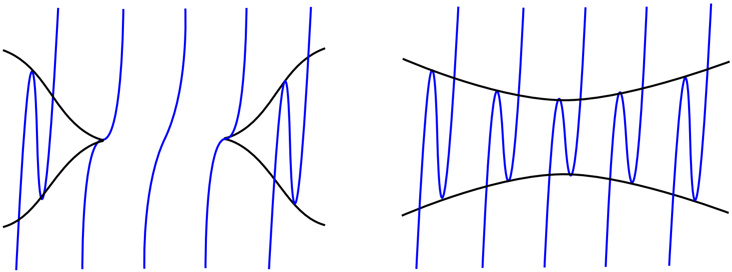
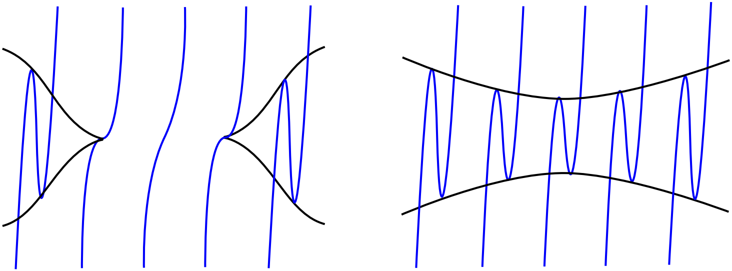
- **4.** Lagrangian concordances induced by Legendrian isotopies and certain Lagrangian handle attachments (compatible with a given one at the negative end), and certain examples of immersed Lagrangian cobordisms (Bourgeois–Sabloff–Traynor '15, Pezzimenti '18).

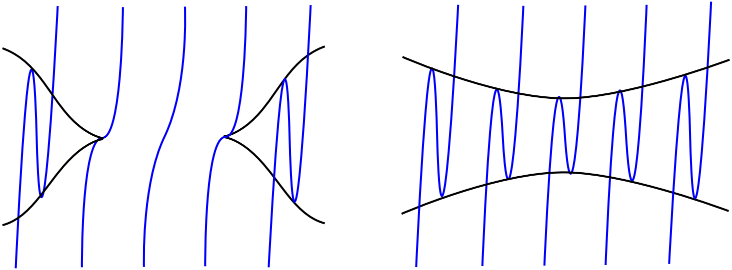
- We explain the homotopical obstruction $L \to \Omega(U/O) \xrightarrow{\sim} \mathbb{Z} \times BO$ from the Morse–Cerf theory perspective.



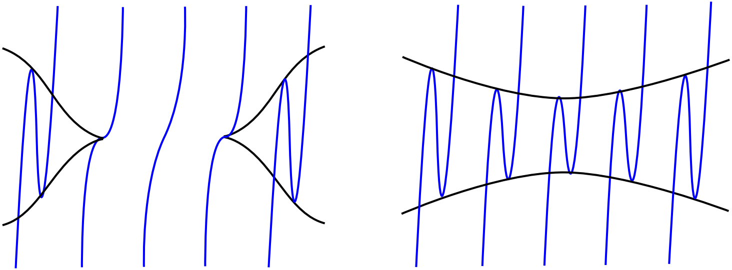
- We explain the homotopical obstruction $L \to \Omega(U/O) \xrightarrow{\sim} \mathbb{Z} \times BO$ from the Morse–Cerf theory perspective.
- Suppose we want to extend generating families on an S^{k-1}-family of cusps to the interior D^k in the front projection when preforming Lagrangian k-handle attachment. We need to introduce a cancelling pair of critical points which is joined to the cancelling pair on the S^{k-1}-family of cusps.



- When $k = 1$, $S^{k-1} = S^0$ and $D^k = D^1$, the obstruction is the Morse index of the pair of critical points near the two cusps, living in $\mathbb{Z}$.



- When $k = 1$, $S^{k-1} = S^0$ and $D^k = D^1$, the obstruction is the Morse index of the pair of critical points near the two cusps, living in $\mathbb{Z}$.
- When $k \geq 2$, Cerf '70 and Hatcher–Wagoner '73 explained that there is a higher obstruction. A cancelling pair of critical points determines a vector space in $\mathbb{R}^\infty$ by taking the negative eigenspace of the Hessian. We thus have an S^{k-1}-family of vector spaces in $\mathbb{R}^\infty$.



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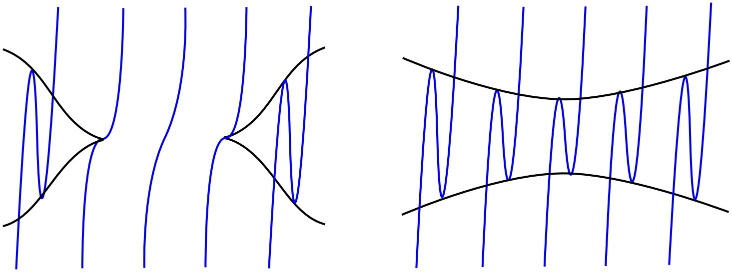
- In order to extend the S^{k-1}-family of cancelling pairs to D^k, the S^{k-1}-family of vector spaces in $\mathbb{R}^\infty$ should be trivial. This is an obstruction in $[S^{k-1}, \mathrm{Gr}(\mathbb{R}^\infty, \mathbb{R}^\infty)] = [S^{k-1}, BO]$.



Extension of constructible sheaves

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Let $\Lambda \subset J^1M \subset T^{*,\infty}(M \times \mathbb{R})$ be a Legendrian. $Sh_\Lambda(M \times \mathbb{R})$ is the dg derived category of sheaves (values in chain complexes) with singular support on Λ. (In particular, such sheaves are locally constant with respect to the stratification of the front projection $\pi(\Lambda) \subset M \times \mathbb{R}$.)

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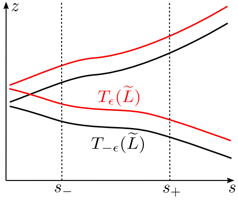
Theorem (L. extension of construtible sheaves)

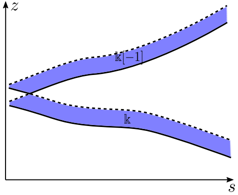
For $L \subset J^1M$ a Legendrian cobordism from Λ_- to Λ_+ with no Reeb chords, a sheaf in $Sh_{\Lambda_-}(M \times \mathbb{R})$ extends to $Sh_L(M \times \mathbb{R}_+ \times \mathbb{R})$ if and only if L is equipped with Maslov data (the Maslov class and relative second Stiefel–Whitney class vanishes) and the given microlocal monodromy data in $Loc(\Lambda_-)$ extends to $Loc(L)$. More precisely, there is an equivalence

$$Sh_{\Lambda_-}(M \times \mathbb{R})_0 \times_{Loc(\Lambda_-)} Loc(L) \simeq Sh_L(M \times \mathbb{R}_+ \times \mathbb{R})_0.$$

Strategy of the proof

- First, starting from a germ of generating families (a microlocal local system) and a generating family (a sheaf) at the negative end that is compatible, we construct a generating family (a sheaf) on the doubling given by L and a small Reeb pushoff of L.





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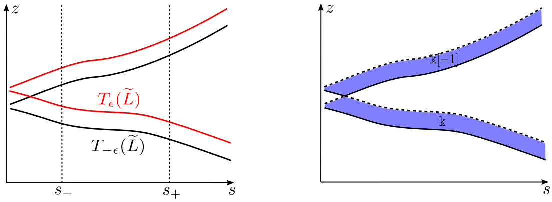
- Second, using a Hamiltonian isotopy, we push the small Reeb pushoff of L to a large Reeb pushoff of L. Generating families (sheaves) are known to propagate.



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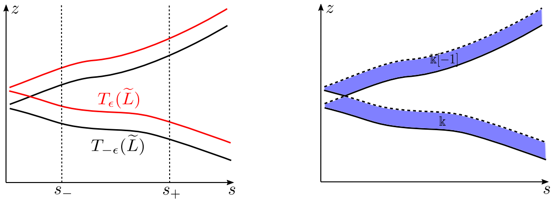
- Second, using a Hamiltonian isotopy, we push the small Reeb pushoff of L to a large Reeb pushoff of L. Generating families (sheaves) are known to propagate.
- Finally, for a sufficiently large Reeb pushoff, the two copies of the front projections of L become separate. We can then cut off the generating family (sheaf) at a certain height.



Thank you!